Determine if the sequence $a_n = \dfrac{\cos^2 n}{2^n}$ converges or diverges. If it converges, find its limit. **SCORE: _____ / 4 PTS**

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

(1) $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ (1) SINCE $0 \le \cos^2 n \le 1$

$\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$ (1)

SO $\lim\limits_{n\to\infty} \dfrac{\cos^2 n}{2^n} = 0$ $\left(\frac{1}{2}\right)$

SEQUENCE CONVERGES $\left(\frac{1}{2}\right)$

Determine if the sequence $a_n = n \sin \frac{1}{n}$ converges or diverges. If it converges, find its limit. **SCORE: _____ / 5 PTS**

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \underbrace{\lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}}}_{1\frac{1}{2}} = \underbrace{\lim_{x \to \infty} \frac{-\frac{1}{x^2} \cos \frac{1}{x}}{-\frac{1}{x^2}}}_{1\frac{1}{2}}$$

$\uparrow$

$\infty \cdot 0$

INDETERMINATE

$$= \cos 0$$

$$= \underbrace{1}_{1\frac{1}{2}} \quad \text{so } \lim_{n \to \infty} n \sin \frac{1}{n} = 1$$

SEQUENCE $\underbrace{\text{CONVERGES}}_{\frac{1}{2}}$

Determine if the sequence defined recursively by $a_1 = 2, \quad a_{n+1} = \frac{1}{2}(6 + a_n)$ is increasing or decreasing. **SCORE: _____ / 4 PTS**

Use mathematical induction to prove your answer. State your conclusion clearly.

$a_2 = \frac{1}{2}(6+2) = 4$

$a_3 = \frac{1}{2}(6+4) = 5$

$a_4 = \frac{1}{2}(6+5) = \frac{11}{2}$

LOOKS LIKE $\{a_n\}$ IS INCREASING

$\frac{1}{2}$ $a_2 > a_1$ SINCE $4 > 2$

$\frac{1}{2}$ IF $a_{k+1} > a_k$ FOR SOME INTEGER $k \geq 1$,

THEN $6 + a_{k+1} > 6 + a_k$

AND $\frac{1}{2}(6 + a_{k+1}) > \frac{1}{2}(6 + a_k)$, 1

SO $a_{k+2} > a_{k+1}$ 1

BY INDUCTION, $a_{n+1} > a_n$ FOR ALL $n \geq 1$

SO $\{a_n\}$ IS INCREASING, 1

Consider the sequence defined recursively by $a_1 = 3, \quad a_{n+1} = \dfrac{a_n}{1+a_n}$. **SCORE: _____ / 3 PTS**

[a] List the first 4 terms of the sequence. Write your final answer as a list.

$$a_2 = \frac{a_1}{1+a_1} = \frac{3}{1+3} = \frac{3}{4}$$

$$a_3 = \frac{a_2}{1+a_2} = \frac{\frac{3}{4}}{1+\frac{3}{4}} \cdot \frac{4}{4} = \frac{3}{7}$$

$$a_4 = \frac{a_3}{1+a_3} = \frac{\frac{3}{7}}{1+\frac{3}{7}} \cdot \frac{7}{7} = \frac{3}{10}$$

$3, \frac{3}{4}, \frac{3}{7}, \frac{3}{10}$ (2) $= \frac{1}{2}$ POINT EACH

[b] Find a formula for the general term a_n of the sequence, assuming that the pattern of the first four terms from [a] continues.

$$a_n = \frac{3}{1+3(n-1)} = \boxed{\frac{3}{3n-2}}$$ (1)

Determine if $\left\{ \dfrac{n^2}{\sqrt{n^3+4}} \right\}$ converges or diverges. If it converges, find its limit. **SCORE: _____ / 4 PTS**

Justify your answer using proper algebra and/or calculus. State your conclusion clearly.

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+4}} = \underbrace{\lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n}+\frac{4}{n^4}}}}_{1\frac{1}{2}} \underbrace{= \infty}_{1} \quad \text{SINCE } \underbrace{\frac{1}{n}+\frac{4}{n^4} \longrightarrow 0}_{1}$$

AS $n \to \infty$

OR

OR

OR

$$= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\sqrt{1+\frac{4}{n^3}}} = \infty$$

SINCE $n^{\frac{1}{2}} \longrightarrow \infty$

AND $1+\frac{4}{n^3} \longrightarrow 1$

AS $n \to \infty$

SEQUENCE DIVERGES $\frac{1}{2}$